Criteria of choosing strategy in games against nature

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2. 08. 2005

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   - Competition on telecomunications services markets
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Introduction

Subject of Game Theory

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  - The result of a game is determined by combinaton of strategies choosen by each of players.

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- Game is a situation, where:
  - There are at least two players.
  - Each player has a set of admissible strategies, that specify the way of his playing.
  - The result of a game is determined by combinaton of strategies choosen by each of players.
  - Every result of game coresponds to payoffs of each player, that can be expressed quantitatively.

Illustration of strategy and payoff concept.

Table: Payoff matrix for two players.

	b_1	b_2	b_3	b_4
a_1			$\vdots$	
a_2	$\ldots\ldots\ldots$	$\ldots\ldots\ldots$	$[V_3^A(a_2), V_2^B(b_3)]$	$\ldots\ldots\ldots$
a_3			$\vdots$	
a_4			$\vdots$	

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a_1			$\vdots$	
a_2	....	....	$[V_3^A(a_2), V_2^B(b_3)]$	....
a_3			$\vdots$	
a_4			$\vdots$	

Game against nature is a situation when player A doesn't know the payoffs of player B.

Competition on telecomunications services markets

- On telecommunications services markets each player aspires to realize of its own business aims, such as:
  - maximization of profits
  - maximization of market share
  - minimization of costs
  - optimization of trafic distribution

Competition on telecomunications services markets

- On telecommunications services markets each player aspires to realize of its own business aims, such as:
  - maximization of profits
  - maximization of market share
  - minimization of costs
  - optimization of trafic distribution
- In general terms we can consider such situations as maximization of payoff function based on demand and/or cost models with prices of provided services treated as decision variables.

Definition of a market game

- Single-criteria market game – the poin g in two-dimensional space $g \in [M, V]$, where:
  - M – stands for a given telecommunication market (local telephony market, internet servieces market, leased lines market, trafic termination market, etc.)
  - V – stands for criterion of decision evaluation – payoff function (profits, market share, costs, trafic distribution, etc.)

e.g. g - game for profits on the local telephony market.

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- Player – telecommunication undertaking as operator or service provider.

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  e.g. g - game for profits on the local telephony market.
- Player – telecommunication undertaking as operator or service provider.
- Strategy – set of telecommunications services and the corespond set of prices

Game against nature on the telecommunications markets

- An A player plays in game against nature, when:
  - other players play in games, for which payoff functions are based on a costs model, that are unknown to an A player, or
  - an A player does not know which game other players are playing in.

Game against nature on the telecommunications markets

- An *A* player plays in game against nature, when:
  - other players play in games, for which payoff functions are based on a costs model, that are unknown to an *A* player, or
  - an *A* player does not know which game other players are playing in.
- The aim is to support market players in the setting prices process on detail and wholesale markets (prices for end users and for interconnected undertakings)

Criteria of choosing strategy in games against nature

Commonly known criteria

- Wald's criterion
- Optimistic criterion
- Hurwicz's criterion
- Laplace's criterion
- Savage's criterion

Commonly known criteria

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- Laplace's criterion
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Wald's criterion

Wald's criterion is prepared for the worst situation – player B chooses the strategy, which for given strategy of player A, leads to the worst outcome for player A.

Table: Pay off matrix of player A.

	b_1	b_2	b_3	b_4
a_1	2	2	0	1
a_2	1	1	1	1
a_3	0	4	0	0
a_4	1	3	0	0

Wald's criterion

Wald's criterion chooses the strategy with the largest minimum outcome:

$$\max\{\min_j V_j(a_i) : i \in \mathcal{I}_A\}.$$

Commonly known criteria

- Wald's criterion
- Optimistic criterion
- Hurwicz's criterion
- Laplace's criterion
- Savage's criterion

Optimistic criterion

Optimistic criterion was prepared for the best situation — player B chooses the strategy, which for given strategy of player A, leads to the best outcome for player A.

Table: Pay off matrix of player A.

	b_1	b_2	b_3	b_4
a_1	2	2	0	1
a_2	1	1	1	1
a_3	0	**4**	0	0
a_4	1	3	0	0

Optimistic criterion

Optimistic criterion chooses the strategy with the largest maximum outcome:

$$\max\{\max_j V_j(a_i) : i \in \mathcal{I}_A\}.$$

Commonly known criteria

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- Laplace's criterion
- Savage's criterion

Hurwicz's criterion

Hurwicz's criterion is the combination of the Wald's criterion and Optimistic criterion with the coefficient of optimism – α.

Hurwicz's criterion

$$\max\{\alpha \cdot \max_j V_j^A(a_i) + (1-\alpha) \cdot \min_j V_j^A(a_i) : i \in \mathcal{I}_A\}.$$

Table: Pay off matrix of player A.

	b_1	b_2	b_3
a_1	10	0	0
a_2	7	2	2
a_3	3	3	3

Hurwicz's criterion indicates on strategy:

- a_3 for $0 \leqslant \alpha \leqslant 0.2$,
- a_2 for $0.2 \leqslant \alpha \leqslant 0.4$,
- a_1 for $0.4 \leqslant \alpha \leqslant 1$.

Commonly known criteria

- Wald's criterion
- Optimistic criterion
- Hurwicz's criterion
- Laplace's criterion
- Savage's criterion

Laplace's criterion

Laplace's criterion chooses the strategy with the highest sum or average value of outcomes.

Table: Pay off matrix of player A.

	b_1	b_2	b_3	b_4
a_1	2	2	0	1
a_2	1	1	1	1
a_3	0	4	0	0
a_4	1	3	0	0

Laplace's criterion

$$\max\{\sum_j V_j(a_i) : i \in \mathcal{I}_A\}.$$

Commonly known criteria

- Wald's criterion
- Optimistic criterion
- Hurwicz's criterion
- Laplace's criterion
- Savage's criterion

Savage's criterion

Savage's criterion is based on the observation, that in some cases a decision maker evaluates the value of obtained outcomes in relation not to its absolute value, but in relation to the highest value of the outcome, which he/she could get, by choosing different strategy, with assuming given strategy of the other player.

Regret function:

$$\widetilde{V}_j^A(a_i) = \max_i V_j^A(a_i) - V_j^A(a_i).$$

Table: Pay off matrix of player A.

	b_1	b_2	b_3	b_4
a_1	**2**	2	0	**1**
a_2	1	1	**1**	1
a_3	0	**4**	0	0
a_4	1	3	0	0

Table: Regret matrix of player A.

	b_1	b_2	b_3	b_4
a_1	0	2	1	0
a_2	1	3	0	0
a_3	2	0	1	1
a_4	1	1	1	1

Savage's criterion

Savage's criterion

Savage's criterion chooses strategy that has the lowest maximum value of regret function:

$$\min\{\max_j \widetilde{V}_j^A(a_i) : i \in \mathcal{I}_A\}.$$

Table: Pay off matrix of player A.

	b_1	b_2	b_3	b_4
a_1	2	2	0	1
a_2	1	1	1	1
a_3	0	4	0	0
a_4	1	3	0	0

Table: Regret matrix of player A.

	b_1	b_2	b_3	b_4
a_1	0	**2**	1	0
a_2	1	**3**	0	0
a_3	**2**	0	1	1
a_4	1	1	1	1

New criteria

- Criterion NHOT
- Criterion SHOT
- Criterion EOT
- Criterion ERT
- Criterion TEO
- Criterion TER

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Criterion NHO

Criterion NHO

- Criterion of maximizing of the Numbers of the Highest Outcomes
- Criterion NHO aims at selecting strategy that has the highest numbers of the outcomes with minimum regret (regret function equals zero).

Table: Pay off matrix of player A.

	b_1	b_2	b_3	b_4
a_1	**2**	2	0	0
a_2	1	1	**1**	**1**
a_3	0	**4**	0	0
a_4	1	3	0	0

Table: Regret matrix of player A.

	b_1	b_2	b_3	b_4
a_1	**0**	2	1	1
a_2	1	3	**0**	**0**
a_3	2	**0**	1	1
a_4	1	1	1	1

Criterion NHO

Criterion NHO

$$\max\{\sum_j \Phi(\widetilde{V}_j^A(a_i)) : i \in \mathcal{I}_A\}$$

where:

$$\Phi(x) = \begin{cases} 1 & for \quad x \leq 0 \\ 0 & for \quad x > 0 \end{cases}$$

$\widetilde{V}_j^A(a_i)$ – regret function

$$\Phi(x) = \frac{sign(-x) + 1}{2}$$

Table: Pay off matrix of player A.

	b_1	b_2	b_3	b_4
a_1	**2**	2	0	0
a_2	1	1	**1**	**1**
a_3	0	**4**	0	0
a_4	1	3	0	0

Table: Regret matrix of player A.

	b_1	b_2	b_3	b_4
a_1	**0**	2	1	1
a_2	1	3	**0**	**0**
a_3	2	**0**	1	1
a_4	1	1	1	1

Criterion NHOT

Criterion NHOT

- Criterion of maximizing of the Numbers of the Highest Outcomes with the Threshold of recognition
- Criterion NHOT aims at selecting strategy that has the highest numbers of the outcomes with regret smaller than or equals to the value of assumed threshold – ρ.

Table: Pay off matrix of player A.

	b_1	b_2	b_3	b_4
a_1	**8**	2	**10**	1
a_2	7	10	9	**10**
a_3	0	**11**	0	0
a_4	1	3	0	0

Table: Regret matrix of player A.

	b_1	b_2	b_3	b_4
a_1	**0**	9	**0**	9
a_2	1	1	1	**0**
a_3	8	**0**	10	1
a_4	7	8	10	1

Criterion NHOT

Criterion NHOT will be formulated with distinction on two cases: absolute and relative threshold - ρ.

Criterion NHOT: absolute threshold - ρ

$$\max\{\sum_j \cdot \Phi(\widetilde{V}_j^A(a_i), \rho) : i \in \mathcal{I}_A\},$$

where:

$$\Phi(x, \rho) = \begin{cases} 1 & \mathit{for} \quad x \leq \rho \\ 0 & \mathit{for} \quad x > \rho \end{cases}$$

Criterion NHOT: relative threshold - ρ

$$\max\{\sum_j \cdot \Phi(\widetilde{V}_j^A(a_i), \rho, V_{j\,\max}^A) : i \in \mathcal{I}_A\}$$

where:

$$\Phi(x, \rho, x_{\max}) = \begin{cases} 1 & \mathit{for} \quad \frac{x}{x_{\max}} \leq \rho \\ 0 & \mathit{for} \quad \frac{x}{x_{\max}} > \rho \end{cases}$$

Criterion NHOT

Absolute threshold ρ

$$\Phi(x, \rho) = \frac{sign(\rho - x) + 1}{2}$$

Relative threshold ρ

$$\Phi(x, \rho, x_{\max}) = \frac{sign(\rho - \frac{x}{x_{\max}}) + 1}{2}$$

Example.

Absolute treshold $\rho = 2$

Table: Pay off matrix of player A.

	b_1	b_2	b_3	b_4
a_1	**8**	2	**10**	1
a_2	7	10	9	**10**
a_3	0	**11**	0	0
a_4	1	3	0	0

Table: Regret matrix of player A.

	b_1	b_2	b_3	b_4
a_1	**0**	9	**0**	9
a_2	1	1	1	**0**
a_3	8	**0**	10	1
a_4	7	8	10	1

New criteria

- Criterion NHOT
- Criterion SHOT
- Criterion EOT
- Criterion ERT
- Criterion TEO
- Criterion TER

Criterion SHOT

Criterion SHOT

- Criterion of maximizing of the Sum of the Highest Outcomes with the Threshold of recognition – ρ.
- Criterion SHOT aims at selecting strategy that has the highest sum of the outcomes with regret smaller than or equals to the value of assumed threshold.
- Combination of NHOT and Laplace

Table: Pay off matrix of player A.

	b_1	b_2	b_3	b_4
a_1	**2**	2	**1**	0
a_2	1	0	**1**	**1**
a_3	0	**4**	0	0
a_4	1	3	0	0

Table: Regret matrix of player A.

	b_1	b_2	b_3	b_4
a_1	**0**	2	**0**	1
a_2	1	4	**0**	**0**
a_3	2	**0**	1	1
a_4	1	1	1	1

Criterion SHOT

Criterion SHOT will be formulated with distinction on two cases: absolute and relative threshold - ρ.

Criterion SHOT: absolute threshold - ρ

$$\max\{\sum_j V_j^A(a_i) \cdot \Phi(\widetilde{V}_j^A(a_i), \rho) : i \in \mathcal{I}_A\}$$

where:

$$\Phi(x, \rho) = \begin{cases} 1 & for \quad x \leq \rho \\ 0 & for \quad x > \rho \end{cases}$$

Criterion SHOT: relative threshold - ρ

$$\max\{\sum_j V_j^A(a_i) \cdot \Phi(\widetilde{V}_j^A(a_i), \rho, V_{j\,\mathrm{max}}^A) : i \in \mathcal{I}_A\} \quad (1)$$

where:

$$\Phi(x, \rho, x_{\mathrm{max}}) = \begin{cases} 1 & for \quad \frac{x}{x_{\mathrm{max}}} \leq \rho \\ 0 & for \quad \frac{x}{x_{\mathrm{max}}} > \rho \end{cases}$$

New criteria

- Criterion NHOT
- Criterion SHOT
- Criterion EOT
- Criterion ERT
- Criterion TEO
- Criterion TER

Criterion EOT

Criterion EOT

- Criterion of maximizing of the Expected value of Outcoms with the Threshold of recognition
- Criterion EOT aims at selecting strategy that has the highest sum of the outcomes which value is higher than or equals to the value of assumed threshold – v.

Getting of which value of pay off function can be treated as success?

Table: Pay off matrix of player A.

	b_1	b_2	b_3	b_4
a_1	3	3	1	1
a_2	2	2	2	2
a_3	4	2	1	1
a_4	5	2	1	0

The sum of payoffs for each strategy a_i equals 8.

Criterion EOT

Criterion EOT

$$\max\{\sum_j V_j^A(a_i)\cdot\Psi(V_j^A(a_i),v) : i\in\mathcal{I}_A\}$$

where:

$$\Psi(x,v)=\begin{cases}1 & \textit{for} \quad x\geq v\\ 0 & \textit{for} \quad x<v\end{cases}$$

v – the value of threshold.

$$\Psi(x,v)=\frac{sign(x-v)+1}{2}$$

Table: Pay off matrix of player A.

	b_1	b_2	b_3	b_4
a_1	3	3	1	1
a_2	2	2	2	2
a_3	4	2	1	1
a_4	5	2	1	0

Examle.

Threshold $v=2$

New criteria

- Criterion NHOT
- Criterion SHOT
- Criterion EOT
- Criterion ERT
- Criterion TEO
- Criterion TER

Criterion ERT

- Criterion of minimizing of the Expected value of Regret with the Threshold of recognition – v.
- Criterion ERT aims at selecting strategy that has the lowest sum of the regret function, which value is higher than or equals to the value of assumed threshold – v.

Criterion ERT

$$\min\{\sum_j \widetilde{V}_j^A(a_i) \cdot \Psi(\widetilde{V}_j^A(a_i), v) : i \in \mathcal{I}_A\}$$

where:

$$\Psi(x, v) = \begin{cases} 1 & for \quad x \geq v \\ 0 & for \quad x < v \end{cases}$$

$\widetilde{V}_j^A(a_i)$ – regret function, v – the value of threshold.

Short comparison

Criterion	Result
EOT	maximizes the sum of outcomes which values can be treated as a winnings (higher than or equal to the value of assumed threshold)
ERT	Maximizes the sum of outcomes for which there is a recognized regret (higher than or equal to the value of assumed threshold)
SHOT	Maximizes the sum of outcomes for which there is not a recognized regret (smaller than or equal to the value of assumed threshold)

New criteria

- Criterion NHOT
- Criterion SHOT
- Criterion EOT
- Criterion ERT
- Criterion TEO
- Criterion TER

Criterion TEO

- Criterion of maximizing of the Treshold Expected value of Outcomes
- Criterion TEO aims at selecting strategy that has the highest sum of the outcomes which value is higher than the a priori unknown value of threshold (EOT with unknown threshold).

Criterion TEO

$$\max\{\sum_{k}\sum_{j} V_j^A(a_i)\cdot\Psi(V_j^A(a_i), k\cdot v) : i \in \mathcal{I}_A\}.$$

where:

$$\Psi(x,v) = \begin{cases} 1 & \mathit{for} \quad x \geq v \\ 0 & \mathit{for} \quad x < v \end{cases}$$

and v is constant, and $k \cdot v$ express the value of threshold.

Criterion TEO

Table: Pay off matrix of player A.

	b_1	b_2	b_3
a_1	1	10	4
a_2	5	5	5
a_3	2	7	6

The sum of outcomes for each strategy equals 15.

Criterion TEO chooses here the strategy a_1 as having the highest value of the sum $(= 117)$.

Table: Table of sums of the outcoms of player A for individual strategies, with changing value of the treshold $k \cdot v$.

$k \cdot v$	a_1	a_2	a_3
1	15	15	15
2	14	15	15
3	14	15	13
4	14	15	13
5	10	15	13
6	10	0	13
7	10	0	7
8	10	0	0
9	10	0	0
10	10	0	0
Sum	**117**	75	89

New criteria

- Criterion NHOT
- Criterion SHOT
- Criterion EOT
- Criterion ERT
- Criterion TEO
- Criterion TER

Criterion TER

- Criterion of minimizing of the Treshold Expected value of Regret
- Criterion TER aims at selecting strategy that has the lowest sum of the regret functions which value is higher than the a priori unknown value of threshold (ERT with unknown threshold).

Criterion TER

$$\min\{\sum_k \sum_j \widetilde{V}_j^A(a_i) \cdot \Psi(\widetilde{V}_j^A(a_i), k \cdot v) : i \in \mathcal{I}_A\}$$

where:

$$\Psi(x, v) = \begin{cases} 1 & for \quad x \geq v \\ 0 & for \quad x < v \end{cases}$$

and $\widetilde{V}_j^A(a_i)$ – regret function, v is constant, and $k \cdot v$ express the value of threshold.

Criterion TER

Table: Regret matrix of player A.

$k \cdot v$	b_1	b_2	b_3
a_1	4	0	2
a_2	0	5	1
a_3	3	3	0

Criterion TER chooses here the strategy a_3 as having the lowest value of the sum $(= 24)$.

Table: Table of the sums of regrets of player A for individual strategies, with changing value of the treshold $k \cdot v$.

	a_1	a_2	a_3
0	6	6	6
1	6	6	6
2	6	5	6
3	4	5	6
4	4	5	0
5	0	5	0
Sum	26	32	**24**

Criterion TER

We can define criterion TER also with relative threshold:

Criterion TER

$$\min\{\sum_k \sum_j \widetilde{V}_j^A(a_i) \cdot \Psi(\widetilde{V}_j^A(a_i), k \cdot v, V_{j\,\max}^A) : i \in \mathcal{I}_A\}$$

where:

$$\Psi(x, v) = \begin{cases} 1 & \mathit{for} \quad \frac{x}{x_{\max}} \geq v \\ 0 & \mathit{for} \quad \frac{x}{x_{\max}} < v \end{cases}$$

and $\widetilde{V}_j^A(a_i)$ – regret function, $V_{j\,\max}^A$ – the highest outcome in column j, v is constant, and $k \cdot v$ express the value of threshold.

Multicriteria analysis

- Regularisation of ambiguous solusions – reduction of the size of the set of results, ambiguous in the sens of one criterion (e.g. Wald), by useing other criterion (e.g. Savage)

Multicriteria analysis

- Regularisation of ambiguous solusions – reduction of the size of the set of results, ambiguous in the sens of one criterion (e.g. Wald), by useing other criterion (e.g. Savage)
- Reference Point Method

Multicriteria analysis

Regularisation of ambiguous solusions

This is an exaple of multicriteria optimisation, where:

- criteria of chooseing strategy (e.g. Wald, Savage, etc.) are treated as criteria of evaluation of solusions
- scalarized aggregation is made as lexicographic optimisation

Multicriteria analysis

Reference Point Method

- criteria of chooseing strategy (e.g. Wald, Savage, etc.) are treated as criteria of evaluation of solusions
- reference point reflects acceptable and satisying values of such criteria.

Thank you for your attention.

Sylwester Laskowski